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Compositional analysis for linear systems

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ABSTRACT

Compositional analysis techniques such as assume-guarantee reasoning are frequently used in computer science to validate the design of complex process models. Since many engineering systems are built modularly from interconnections of components, the resulting mathematical models can be arbitrarily complex, which makes their analysis equally challenging. This paper presents a framework of how to apply compositional and assume-guarantee reasoning to linear time-invariant (LTI) systems. A key tool are simulation relations which are used to relate systems models with their specifications as well as to determine abstractions of given system behaviors. First, complex systems defined by standard feedback interconnections are considered. Parallel composition of LTI systems, the second type of interconnections, introduces algebraic constraints but allows for decomposition of a global specification.

1. Introduction

Many engineering applications such as chemical plants, mechatronic systems or discretized PDE models are described by ever more complex mathematical models with a large number of state components. Moreover, a global process model is often built from modular subprocesses which are interconnected through their input–output pairs. In a decentralized control scheme, for example, a global control target is achieved by a network of locally controlled subsystems. The analysis of such complex system models is challenging. A similar problem occurs in formal verification, an area of computer science where implementations of computer programs are checked for correctness. This led to the development of concepts that reduce the inherent complexity of verification tasks, see for example [1,2]. Milner [3] introduced the concept of simulation relations to compare the stepwise behavior of two transition systems. To check whether a given program matches the desired specification, a simulation relation is sought to be constructed relating the transition system expressing the program with the transition system representing the property. Apart from verifying properties of implemented process models, simulation relations can also serve as a tool to abstract a given system by a lower order one. Simulation relations have been adapted to dynamical systems using geometric control theory, see in particular [4–6]. Abstractions of dynamical systems were discussed in [7], which can also be used to reduce the complexity of interconnected system models. An abstract treatment of bisimulation relations to solve the controller synthesis problem using category theory can be found in [8]. Compositional and assume-guarantee reasoning [9–11] provides strategies to decompose a verification task for a labeled transition system into several tasks involving individual components or components restricted to a specific environment. First extensions were achieved for hybrid systems in [11]. More recently, compositional reasoning has been investigated for linear [12] and hybrid feedback control systems [13] emphasizing the differential equation description instead of their solution set. This paper extends the latter methodology to more general types of interconnections, see [14] for a preliminary version. At first, feedback interconnections are considered. Compositionality of feedback interconnections is proven as well as the validity of both non circular and circular assume-guarantee reasoning, which is illustrated with an example from circuit theory. In the second part, parallel composition of linear systems is introduced. The resulting algebraic constraints on the system variables are characteristic for models of physical processes. The analysis of parallel composition includes a decomposition strategy for a given global specification, i.e., how a proof obligation for the overall specification can be reduced to a number of less complex proof obligations each involving a sub-specification.

2. Preliminaries

Consider the class of linear continuous-time systems

$$\Sigma_i : \quad \begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i + G_i e_i + L_i d_i \\ y_i &= C_i x_i \\ z_i &= H_i x_i. \end{aligned} \tag{1}$$

All variables belong to finite dimensional vector spaces, $x_i \in \mathcal{X}_i, u_i \in \mathcal{U}_i, e_i \in \mathcal{E}_i, d_i \in \mathcal{D}_i, y_i \in \mathcal{Y}_i, z_i \in \mathcal{Z}_i$. The temporal

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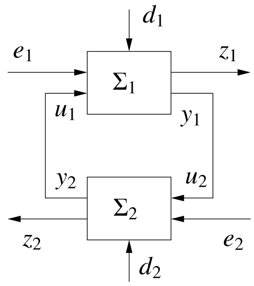

Fig. 1. Interconnection $\Sigma_1 \| \Sigma_2$.

evolution of all system variables is characterized by functions of an appropriate function class, e. g. $\mathcal{C}^\infty$. The variables u_i and y_i are used for interconnections, e_i and z_i are inputs and outputs to specify the performance, while d_i represents a disturbance.

Remark 1. Disturbance inputs are often used to model uncertainties, e. g. parameter uncertainties or unmodeled dynamics. As proposed in [5,6], a system of the form (1) can *abstract* a linear system of higher state space dimension. More concretely, a system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2}$$

$$y = \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

can be abstracted by the lower order model

$$\begin{aligned} \dot{z} &= A_{11} z + A_{12} d \\ y &= C_1 z \end{aligned} \tag{3}$$

with d a disturbance, in the sense that (2) is simulated by (3).

Definition 1. The feedback interconnection $\|$ of two linear continuous-time systems $\Sigma_i,\ i = 1, 2$, is defined as

$$u_2 = y_1, \qquad u_1 = y_2. \tag{4}$$

The dynamics of the interconnected system $\Sigma_1 \| \Sigma_2$ are then given by (see Fig. 1)

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_1 & B_1 C_2 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \\ &\quad + \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \\ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \end{aligned} \tag{5}$$

We recall the main definitions and results of simulation theory for linear systems from [6,5].

Definition 2. A simulation relation S of Σ_1 by Σ_2 is a linear subspace $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ with the following property: For any $(x_{10}, x_{20}) \in S$, any joint input function $e_1(\cdot) = e_2(\cdot) = e$, any joint interconnection input $u_1(\cdot) = u_2(\cdot) = u(\cdot)$ and any disturbance function $d_1(\cdot)$ there should exist a disturbance $d_2(\cdot)$ such that the resulting state trajectories $x_i(.)$ with $x_i(0) = x_{i0},\ i = 1, 2$, satisfy

$$\begin{aligned} &\text{(i) } (x_1(t), x_2(t)) \in S, \quad \forall t \geq 0 \\ &\text{(ii) } H_1 x_1(t) = H_2 x_2(t), \quad \forall t \geq 0 \\ &\text{(iii) } C_1 x_1(t) = C_2 x_2(t), \quad \forall t \geq 0. \end{aligned} \tag{6}$$

Σ_1 is simulated by Σ_2, denoted by $\Sigma_1 \preccurlyeq \Sigma_2$, if there exists a simulation relation S fulfilling $\Pi_1 S = \mathcal{X}_1$ with $\Pi_1 : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{X}_1$ the canonical projection from $\mathcal{X}_1 \times \mathcal{X}_2$ to $\mathcal{X}_1$. In this case, S is called a *full simulation relation.*

If in addition $S^{-1} := \{(x_2, x_1) \mid (x_1, x_2) \in S\}$ defines a simulation relation of Σ_2 by Σ_1, then $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a *bisimulation relation* between Σ_1 and Σ_2. Moreover, if $\Pi_i S = \mathcal{X}_i,\ i = 1, 2$, then S is called a full bisimulation relation and Σ_1 and Σ_2 are called *bisimilar*, denoted by $\Sigma_1 \approx \Sigma_2$.

Proposition 1. *A subspace $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a simulation relation of Σ_1 by Σ_2 if and only if for all $(x_1, x_2) \in S$, all $u \in \mathcal{U}$ and all $e \in \mathcal{E}$ the following holds:*

$$\begin{aligned} &\text{(i): } \forall\, d_1 \in \mathcal{D}_1\ \exists\, d_2 \in \mathcal{D}_2 : \begin{bmatrix} A_1 x_1 + B_1 u + G_1 e + L_1 d_1 \\ A_2 x_2 + B_2 u + G_2 e + L_2 d_2 \end{bmatrix} \in S \\ &\text{(ii): } H_1 x_1 = H_2 x_2 \\ &\text{(iii): } C_1 x_1 = C_2 x_2. \end{aligned}$$

Theorem 1. *A linear subspace $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a simulation relation of Σ_1 by Σ_2 if and only if the following holds:*

1. $\operatorname{im} \begin{bmatrix} G_1 & B_1 \\ G_2 & B_2 \end{bmatrix} + \operatorname{im} \begin{bmatrix} L_1 \\ 0 \end{bmatrix} \subset S + \operatorname{im} \begin{bmatrix} 0 \\ L_2 \end{bmatrix}$
2. $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} S \subset S + \operatorname{im} \begin{bmatrix} 0 \\ L_2 \end{bmatrix}$
3. $S \subset \ker \begin{bmatrix} H_1 & -H_2 \\ C_1 & -C_2 \end{bmatrix}$.

Finally, simulation relations as defined above retain two important properties of their counterparts for labeled transition systems.

Proposition 2. *Simulation relations $\preccurlyeq$ are preorders, i.e. they are reflexive and transitive.*

Proof. Consider linear systems $\Sigma_i,\ i \in \{1, 2, 3\}$, of the form (1). Reflexivity: The relation $S = \{(x_1, x_1) \mid x_1 \in \mathcal{X}_1\}$ fulfils conditions (i) and (ii) of Definition 2 and therefore defines a full simulation relation of Σ_1 by Σ_1.

Transitivity: Assume S_1 defines a full simulation relation of Σ_1 by Σ_2 and S_2 of Σ_2 by Σ_3. Then $S_{12} = \{(x_1, x_3) \mid \exists x_2 : (x_1, x_2) \in S_1, (x_2, x_3) \in S_2\}$ defines a full simulation relation of Σ_1 by Σ_3. □

Proposition 3. *For any two linear systems Σ_P and Σ_Q,*

$$\Sigma_P \| \Sigma_Q \approx \Sigma_Q \| \Sigma_P \tag{7}$$

after permuting the state vectors of Σ_P and Σ_Q.

Proof. The relation

$$\begin{aligned} S = \{&((x_P, x_Q), (\bar{x}_P, \bar{x}_Q)) \mid (x_P, x_Q) \in \Sigma_P \| \Sigma_Q, \\ &(\bar{x}_Q, \bar{x}_P) \in \Sigma_Q \| \Sigma_P, x_P = \bar{x}_P, x_Q = \bar{x}_Q\} \end{aligned}$$

is a bisimulation relation between $\Sigma_P \| \Sigma_Q$ and $\Sigma_Q \| \Sigma_P$. □

3. Compositional and assume-guarantee reasoning for linear systems

Consider a complex linear plant system Σ_P which we assume to be given in the form of interconnected subsystems $\Sigma_{P_i}, i = 1, \ldots, N$, that is $\Sigma_P = \Sigma_1 \| \ldots \| \Sigma_N$. We want to check whether Σ_P has the desired behavior specified by Σ_Q which again we assume to be given in form of interconnected sub-specifications Σ_{Q_i}, $\Sigma_Q = \Sigma_{Q_1} \| \ldots \| \Sigma_{Q_N}$. For the sake of clarity, we will restrict ourselves to interconnections of two subsystems only. However, the compositional techniques described in the following can be generalized to an arbitrary number of subsystems thanks to their modular structure. Using simulation relations, this verification task can be expressed as

$$\Sigma_{P_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2}. \tag{8}$$

In order to reduce the complexity of the proof obligation, (8) will be decomposed into two less complex subtasks.

3.1. Compositional reasoning

We start with the main pillar for compositional analysis.

Theorem 2. *For any given linear systems $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2\}$, the compositionality property*

$$\left.\begin{array}{l}\Sigma_{P_1} \preccurlyeq \Sigma_{Q_1}\\ \Sigma_{P_2} \preccurlyeq \Sigma_{Q_2}\end{array}\right\} \Longrightarrow \Sigma_{P_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2} \tag{9}$$

holds.

Proof. Let $S_i,\ i = 1, 2$, denote the full simulation relations of Σ_{P_i} by Σ_{Q_i}. Construct the relation

$$S = \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid (x_{P_1}, x_{Q_1}) \in S_1, (x_{P_2}, x_{Q_2}) \in S_2\}. \tag{10}$$

Then for every $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$, every joint input $e_{P_i} = e_{Q_i} = e_i$, $i = 1, 2$ and every disturbance d_{P_1}, d_{P_2}, there exist disturbances d_{Q_1}, d_{Q_2} such that

$$\begin{bmatrix} A_{P_1}x_{P_1} + B_{P_1}C_{P_2}x_{P_2} + G_{P_1}e_1 + L_{P_1}d_{P_1} \\ A_{Q_1}x_{Q_1} + B_{Q_1}C_{Q_2}x_{Q_2} + G_{Q_1}e_1 + L_{Q_1}d_{Q_1} \end{bmatrix} \in S_1$$

and

$$\begin{bmatrix} A_{P_2}x_{P_2} + B_{P_2}C_{P_1}x_{P_1} + G_{P_2}e_2 + L_{P_2}d_{P_2} \\ A_{Q_2}x_{Q_2} + B_{Q_2}C_{Q_1}x_{Q_1} + G_{Q_2}e_2 + L_{Q_2}d_{Q_2} \end{bmatrix} \in S_2$$

whilst $H_{P_i}x_{P_i} = H_{Q_i}x_{Q_i}$, since $C_{P_i}x_{P_i} = C_{Q_i}x_{Q_i}$ for all $(x_{P_i}, x_{Q_i}) \in S_i$. Moreover, S as defined in (10) is in fact the product of the simulation relations S_1 and S_2 after reordering the vectors x_{Q_1} and x_{P_2}. Since $\Pi_{P_1}S_1 = \mathcal{X}_1$ and $\Pi_{P_2}S_2 = \mathcal{X}_2$, i.e. S_1 and S_2 are full, $\Pi_{P_1P_2}S = \mathcal{X}_1 \times \mathcal{X}_2$ and therefore S is full. □

Remark 2. The converse implication in general does not hold. Take as a counterexample the following systems

$$\Sigma_{P_1} : \dot{x}_{P_1} = 2u_{P_1} + e_{P_1} \qquad \Sigma_{P_2} : \dot{x}_{P_2} = u_{P_2} + e_{P_2}$$
$$\Sigma_{Q_i} : \dot{x}_{Q_i} = u_{Q_i} + e_{Q_i}$$
$$y_{P_1} = z_{P_1} = x_{P_1} \qquad y_{P_2} = \frac{1}{2}x_{P_2} \qquad y_{Q_i} = z_{Q_i} = x_{Q_i}$$
$$z_{P_2} = x_{P_2}.$$

Then there exists a simulation relation S of $\Sigma_{P_1} \| \Sigma_{P_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$, namely

$$S = \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid x_{P_1} = x_{Q_1}, x_{P_2} = x_{Q_2}\}$$

since the state space descriptions of $\Sigma_{P_1} \| \Sigma_{P_2}$ and $\Sigma_{Q_1} \| \Sigma_{Q_2}$ are identical. However, there do not exist any simulation relations of Σ_{P_1} by Σ_{Q_1} nor of Σ_{P_2} by Σ_{Q_2} since for the former,

$$\operatorname{im}\begin{bmatrix} B_{P_1} \\ B_{Q_1} \end{bmatrix} = \operatorname{im}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \not\subseteq \ker\begin{bmatrix} C_{P_1} & -C_{Q_1} \\ H_{P_1} & -H_{Q_1} \end{bmatrix} = \operatorname{im}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and for the latter,

$$\operatorname{im}\begin{bmatrix} B_{P_2} \\ B_{Q_2} \end{bmatrix} = \operatorname{im}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \not\subseteq \ker\begin{bmatrix} C_{P_2} & -C_{Q_2} \\ H_{P_2} & -H_{Q_2} \end{bmatrix} = \operatorname{im}\begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We note that as a special case of compositionality, *invariance under composition* also holds:

$$\forall \Sigma_{Q_2} : \Sigma_{P_1} \preccurlyeq \Sigma_{Q_1} \Longrightarrow \Sigma_{P_1} \| \Sigma_{Q_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2}. \tag{11}$$

In fact, since the interconnection $\|$ is commutative, compositionality and invariance under composition are equivalent.

3.2. Assume-guarantee reasoning

In the case that one or more of the components $\Sigma_{P_i},\ i = 1, 2$, do not fulfil their sub-specification Σ_{Q_i} directly, compositional reasoning cannot be applied to simplify the verification task (8). However, restricting the respective component by interconnecting it to a suitable subsystem makes it still possible to derive alternative deduction schemes of lower complexity. In the following, we present two types of assume-guarantee reasoning rules based on this principle. In the fist one, the assumption $\Sigma_{P_2} \preccurlyeq \Sigma_{Q_2}$ is replaced by $\Sigma_{Q_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2}$, that is, Σ_{P_2} is compared with Σ_{Q_2} while it is already *assumed* that Σ_{P_1} may be replaced by Σ_{Q_1}, and similarly for the second rule.

Theorem 3. *For any given linear systems Σ_i, $i \in \{P_1, P_2, Q_1, Q_2\}$, non circular assume-guarantee reasoning is sound, i.e. the following deduction scheme*

$$\left.\begin{array}{l} S_1 : \Sigma_{P_1} \preccurlyeq \Sigma_{Q_1} \\ S_2 : \Sigma_{Q_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2} \end{array}\right\} \Longrightarrow S : \Sigma_{P_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2} \tag{12}$$

and its symmetric counterpart

$$\left.\begin{array}{l} S_1' : \Sigma_{P_2} \preccurlyeq \Sigma_{Q_2} \\ S_2' : \Sigma_{P_1} \| \Sigma_{Q_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2} \end{array}\right\} \Longrightarrow S : \Sigma_{P_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2} \tag{13}$$

hold.

Proof. The proof only requires the relation $\preccurlyeq$ to be a preorder and the interconnection $\|$ to be invariant under composition. For (12), reflexivity of simulation and invariance under composition yield

$$\left.\begin{array}{l}\Sigma_{P_1} \preccurlyeq \Sigma_{Q_1}\\ \Sigma_{P_2} \preccurlyeq \Sigma_{P_2}\end{array}\right\} \Longrightarrow \Sigma_{P_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{P_2}$$

which due to S_2 and transitivity of simulation yields the desired result,

$$\Sigma_{P_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2}. \tag{14}$$

Exploiting commutativity of the interconnection, the same arguments hold for the other non circular rule,

$$\Sigma_{P_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{P_2} \| \Sigma_{P_1} \preccurlyeq \Sigma_{Q_2} \| \Sigma_{P_1} \preccurlyeq \Sigma_{P_1} \| \Sigma_{Q_2} \preccurlyeq \Sigma_{Q_2} \| \Sigma_{Q_2}. \quad □$$

Example 1. Consider as Σ_{P_1} the LC-circuit in Fig. 2 with two inductors L_1 and L_2, one capacitor C, a voltage source as input u_{P_1} and the current over the capacitor as output y_{P_1}. The control in- and outputs are chosen to be the same as the interconnection variables, $u_{P_1} = e_{P_1}$ and $y_{P_1} = z_{P_1}$, while there are no external disturbances, i.e., d_{P_1} is absent. Then, Σ_{P_1} is given by

$$\Sigma_{P_1} : \begin{bmatrix} \dot{q}_C \\ \dot{\phi}_{L_1} \\ \dot{\phi}_{L_2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{L_1} & \frac{1}{L_2} \\ -\frac{1}{C} & 0 & 0 \\ \frac{1}{C} & 0 & 0 \end{bmatrix} \begin{bmatrix} q_C \\ \phi_{L_1} \\ \phi_{L_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_{P_1} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e_{P_1} \tag{15}$$
$$y_{P_1} = \begin{bmatrix} \frac{1}{C} & 0 & 0 \end{bmatrix} x_{P_1} = z_{P_1}$$

where $x_{P_1} = \begin{bmatrix} q_C & \phi_{L_1} & \phi_{L_2} \end{bmatrix}^T$ denotes the state vector. In the remainder, all the parameter values are set to 1. To stabilize the electrical circuit (15) we apply a simple feedback controller Σ_{P_2},

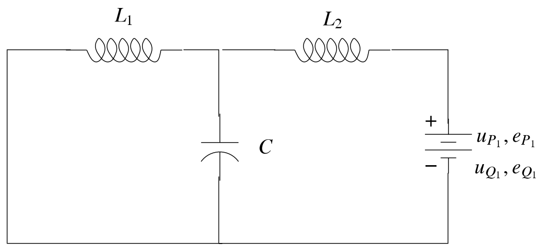

Fig. 2. Σ_{P_1}: *LC*- circuit.

$$\Sigma_{P_2}: \begin{bmatrix}\dot z_1\\ \dot z_2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -5\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix} u_{P_2} \tag{16}$$
$$y_{P_2} = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix}.$$

Observe that $e_{P_2} = z_{P_2} = d_{P_2}$ are all void.

The verification goal is to simulate the 5-dimensional interconnection $\Sigma_{P_1}\|\Sigma_{P_2}$ by a less complex specification $\Sigma_{Q_1}\|\Sigma_{Q_2}$. The components of this specification are given by the *LC*- circuit Σ_{Q_1} as in Fig. 1 and an abstracted controller Σ_{Q_2}. In particular,

$$\Sigma_{Q_1}: \begin{bmatrix}\dot\phi_{Q_1}\\ \dot q_{Q_1}\end{bmatrix} = \begin{bmatrix}0 & -\dfrac{1}{C_{Q_1}}\\ \dfrac{1}{L_{Q_1}} & 0\end{bmatrix}\begin{bmatrix}\phi_{Q_1}\\ q_{Q_1}\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix} u_{Q_1} + \begin{bmatrix}0\\ 1\end{bmatrix} e_{Q_1} + \begin{bmatrix}1\\ 0\end{bmatrix} d_{Q_1} \tag{17}$$
$$y_{Q_1} = \begin{bmatrix}0 & \dfrac{1}{C_{Q_1}}\end{bmatrix} x_{Q_1} = z_{Q_1}$$

where $x_{Q_1} = \begin{bmatrix}\phi_{Q_1} & q_{Q_1}\end{bmatrix}^T$ and all parameter values are again set to 1. The controller Σ_{Q_2} is described by

$$\Sigma_{Q_2}: \dot x_{Q_2} = -5x_{Q_2} + u_{Q_2} + d_{Q_2} \tag{18}$$
$$y_{Q_2} = x_{Q_2}.$$

The first observation is that compositionality is not applicable since there does not exist any simulation relation of Σ_{P_1} by Σ_{Q_1}. The disturbance input d_{Q_1} represents a voltage source which cannot mimic the behavior of the inductor L_2. However, the controller systems Σ_{P_2} and Σ_{Q_2} can be related by means of a full simulation relation S_1',

$$S_1' = \{(z_1, z_2, x_{Q_2}) \mid z_1 = x_{Q_2}\}. \tag{19}$$

Moreover, the interconnected system $\Sigma_{P_1}\|\Sigma_{Q_2}$ can be simulated by $\Sigma_{Q_1}\|\Sigma_{Q_2}$ using the simulation relations

$$S_2' = \{((q_C, \phi_{L_1}, \phi_{L_2}, x_{Q_2}), (x_1, x_2, x_{Q_2}')) \mid x_{Q_2} = x_{Q_2}',$$
$$q_C = x_2, -1/5q_C + 1/5\phi_{L_1} + \phi_{L_2} + x_{Q_2} = x_1\}. \tag{20}$$

By Theorem 3, we can therefore conclude that there exists a full simulation relation S of $\Sigma_{P_1}\|\Sigma_{P_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$, given by

$$S = \{((q_C, \phi_{L_1}, \phi_{L_2}, z_1, z_2), (x_1, x_2, x_{Q_2})) \mid z_2 = x_{Q_2},$$
$$q_C = z_2, q_C - \phi_{L_1} + \phi_{L_2} + x_1 = z_1\}. \tag{21}$$

This shows that it is possible to abstract the behavior of the 5- dimensional controlled electrical circuit by a 3-dimensional electrical circuit with disturbances.

In *circular assume-guarantee* reasoning neither of the relations $\Sigma_{P_i} \preccurlyeq \Sigma_{Q_i}, i = 1, 2$, is assumed unconditionally. Instead, these assumptions are replaced by $S_1: \Sigma_{P_1}\|\Sigma_{Q_2} \preccurlyeq \Sigma_{Q_1}\|\Sigma_{Q_2}$ and $S_2: \Sigma_{Q_1}\|\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|\Sigma_{Q_2}$ in order to prove (8). That is, to guarantee that Σ_{P_1} has property Σ_{Q_1} it is already assumed that Σ_{P_2} fulfils Σ_{Q_2} while conversely Σ_{P_1} is assumed to be replaceable by Σ_{Q_1} to guarantee that Σ_{P_2} has property Σ_{Q_2}. Although for general transition systems circular assume-guarantee reasoning is sound only under additional conditions [11], it *always* holds true for linear systems. The main idea in this proof is to enlarge the simulation relations S_1 and S_2 in a suitable way. Since the proofs of the lemmas and the main theorem are quite technical, we defer them to the Appendix.

Lemma 1. *Given full simulation relations S_i, $i = 1, 2$ of $\Sigma_{P_1}\|\Sigma_{q_2}$ and $\Sigma_{Q_1}\|\Sigma_{P_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$, respectively, and define the following linear subspaces*

$$\begin{aligned}\bar S_1 := \{&(x_{P_1}, \bar x_{Q_2}, x_{Q_1}, -x_{Q_2}) \mid x_{Q_2}, \bar x_{Q_2} \in \ker C_{Q_2} \cap \ker H_{Q_2},\\ &x_{P_1} \in \ker C_{P_1} \cap \ker H_{P_1}, x_{Q_1} \in \ker C_{Q_1} \cap \ker H_{Q_1},\\ &(x_{P_1}, x_{Q_2}, x_{Q_1}, \bar x_{Q_2}) \in S_1\}\end{aligned} \tag{22}$$
$$\begin{aligned}\bar S_2 := \{&(\bar x_{Q_1}, x_{Q_2}, -x_{Q_1}, x_{Q_2}) \mid x_{Q_1},\\ &\bar x_{Q_1} \in \ker C_{Q_1} \cap \ker H_{Q_1}, x_{P_2} \in \ker C_{P_2} \cap \ker H_{P_2},\\ &x_{Q_2} \in \ker C_{Q_2} \cap \ker H_{Q_2}, (x_{Q_1}, x_{Q_1}, \bar x_{Q_1}, x_{Q_2}) \in S_2\}.\end{aligned}$$

Then $S_1 + \bar S_1$ and $S_2 + \bar S_2$ also define full simulation relations of $\Sigma_{P_1}\|\Sigma_{Q_2}$ and $\Sigma_{Q_1}\|\Sigma_{P_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$, respectively.

Proof. The proof can be found in Appendix A. □

Lemma 2. *Given full simulation relations S_1 and S_2 of $\Sigma_{P_1}\|\Sigma_{Q_2}$ and $\Sigma_{Q_1}\|\Sigma_{P_2}$ by $\Sigma_3\|\Sigma_4$, respectively, then also their symmetrized versions*

$$S_1^{\mathrm{sym}} := S_1 + \hat S_1, \qquad S_2^{\mathrm{sym}} := S_2 + \hat S_2 \tag{23}$$

where

$$\hat S_1 := \{(x_{P_1}, \bar x_{Q_2}, x_{Q_1}, x_{Q_2}) \mid (x_{P_1}, x_{Q_2}, x_{Q_1}, \bar x_{Q_2}) \in S_1\} \tag{24}$$
$$\hat S_2 := \{(\bar x_{Q_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid (x_{Q_1}, x_{P_2}, \bar x_{Q_1}, x_{Q_2}) \in S_2\}$$

define full simulation relations of $\Sigma_{P_1}\|\Sigma_{Q_2}$ and $\Sigma_{Q_1}\|\Sigma_{P_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$.

Proof. The proof is very similar to the previous one and can be found in Appendix B. □

Adding additional subspaces $(\bar{.})$ and $(\hat{.})$ to the original relations S_1 and S_2 ensures that the following elements are included in $(S_i + \bar S_i)^{\mathrm{sym}}$.

Lemma 3. *Consider full simulation relation $(S_1 + \bar S_1)^{\mathrm{sym}}$ and $(S_2 + \bar S_2)^{\mathrm{sym}}$ of $\Sigma_{P_1}\|\Sigma_{Q_2}$ and $\Sigma_{Q_1}\|\Sigma_{P_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$ as defined in the previous lemmas. Then for every $x \in \ker C_{Q_2} \cap \ker H_{Q_2}$, $(0, x, 0, x) \in (S_1 + \bar S_1)^{\mathrm{sym}}$ and analogously, for every $y \in \ker C_{Q_1} \cap \ker H_{Q_1}$, $(y, 0, y, 0) \in (S_2 + \bar S_2)^{\mathrm{sym}}$.*

Proof. The proof can be found in Appendix C. □

Using the extended full simulation relations $(S_1 + \bar S_1)^{\mathrm{sym}}$ and $(S_2 + \bar S_2)^{\mathrm{sym}}$ it is possible to construct a full simulation relation S of $\Sigma_{P_1}\|\Sigma_{P_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$.

Theorem 4. *For any given linear systems Σ_i, $i \in \{P_1, P_2, Q_1, Q_2\}$, circular assume-guarantee reasoning is sound, i.e. the deduction*

$$\left.\begin{aligned}S_1 &: \Sigma_{P_1}\|\Sigma_{Q_2} \preccurlyeq \Sigma_{Q_1}\|\Sigma_{Q_2}\\ S_2 &: \Sigma_{Q_1}\|\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|\Sigma_{Q_2}\end{aligned}\right\} \Longrightarrow S: \Sigma_{P_1}\|\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|\Sigma_{Q_2} \tag{25}$$

holds. The full simulation relation S of $\Sigma_{P_1}\|\Sigma_{P_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$ is given by

$$\begin{aligned}S := \{&(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid \exists \bar x_{Q_1}, \bar x_{Q_2} : (x_{P_1}, x_{Q_2}, x_{Q_1}, \bar x_{Q_2})\\ &\in (S_1 + \bar S_1)^{\mathrm{sym}}, (x_{Q_1}, x_{P_2}, \bar x_{Q_1}, x_{Q_2}) \in (S_2 + \bar S_2)^{\mathrm{sym}}\}.\end{aligned} \tag{26}$$

Proof. The proof is included in Appendix D. □

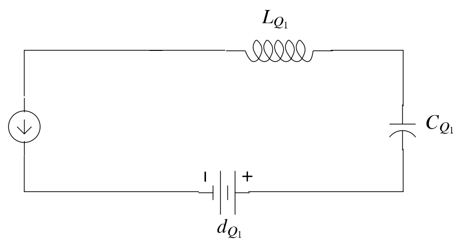

Fig. 3. $\sum_{Q_1}$.

4. Interconnections with algebraic constraints

In many applications, the interconnection of subsystems introduces *algebraic constraints* on the state variables. In this section, we consider *parallel compositions* of linear systems as a specific type of interconnection inducing algebraic constraints on the states. Writing the interconnected constrained system in differential–algebraic form allows to give a geometric characterization of simulation relations for such parallel compositions. Compositional reasoning is developed in conjunction with a decomposition strategy to split a given global specification into an interconnection of possibly lower dimensional specifications.

Definition 3. Given two linear dynamical systems $\Sigma_i,\ i = 1, 2$, of the form

$$\Sigma_i\colon\ \dot{x}_i = A_i x_i + B_i u_i + L_i d_i \tag{27}$$

$$y_i = C_i x_i \tag{28}$$

where $x_i \in \mathcal{X}_i \subseteq \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^p$, $d_i \in \mathcal{D}_i$ and $y_i \in \mathbb{R}^q$.
Then the parallel composition $\Sigma_1 \|_{pc} \Sigma_2$ is given by

$$\Sigma_1 \|_{pc} \Sigma_2\colon \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u + \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \tag{29}$$

$$y = C_1 x_1 = C_2 x_2.$$

Since parallel composition entails the algebraic constraint $C_1 x_1 = C_2 x_2$, depicted in Fig. 3, the Eq. (29) can be rewritten in differential–algebraic form as

$$\Sigma_{12}\colon\ E_{12}\dot{z}_{12} = A_{12} z_{12},\ z_{12} \in \mathcal{Z}_{12} \tag{30}$$

$$w_{12} = C_{12} z_{12}$$

where the matrices E_{12}, A_{12}, C_{12} and the state and output vectors z_{12} and w_{12} are given by

$$z_{12} = \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix}, \qquad A_{12} = \begin{bmatrix} L_1^{\perp} A_1 & 0 & L_1^{\perp} B_1 \\ 0 & L_2^{\perp} A_2 & L_2^{\perp} B_2 \\ C_1 & -C_2 & 0 \end{bmatrix}, \tag{31}$$

$$w_{12} = \begin{bmatrix} y \\ u \end{bmatrix}, \qquad E_{12} = \begin{bmatrix} L_1^{\perp} & 0 & 0 \\ 0 & L_2^{\perp} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad C_{12} = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

respectively, where $L_1^{\perp}$ and $L_2^{\perp}$ are left annihilating matrices of full rank. The set of states and inputs consistent with these constraints is defined by the consistent subspace.

Definition 4. Consider a system Σ_{12} of the form (30). Then the consistent subspace $\mathcal{V}_{12}^{\star}$ for Σ_{12} is the largest subspace $\mathcal{V}_{12} \subset \mathcal{Z}_{12}$ such that

$$A_{12}\mathcal{V}_{12} \subset E_{12}\mathcal{V}_{12}. \tag{32}$$

Furthermore, denote by $\mathcal{W}_{12}^{\star}$ and $\mathcal{U}_{12}^{\star}$ the projections

$$\mathcal{W}_{12}^{\star} = \Pi_{x_1 x_2}\mathcal{V}_{12}^{\star} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid \exists u : \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix} \in \mathcal{V}_{12}^{\star} \right\} \tag{33}$$

$$\mathcal{U}_{12}^{\star} = \Pi_{u}\mathcal{V}_{12}^{\star} = \left\{ u \mid \exists x_1, x_2 : \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix} \in \mathcal{V}_{12}^{\star} \right\}. \tag{34}$$

This allows us to specialize the general definition of simulation relations (Definition 2) to linear systems of the form (29) respectively (30), compare also with [15].

Definition 5. Given two linear systems $\Sigma_i, i = \{P_1P_2, Q_1Q_2\}$ of the form (30) with consistent subspaces $\mathcal{V}_i^{\star}$. Then a subspace $\tilde{S} \subset \mathcal{Z}_{P_1P_2} \times \mathcal{Z}_{Q_1Q_2}$ with $\Pi_{P_1P_2}\tilde{S} \subset \mathcal{V}_{P_1P_2}^{\star}$ is a simulation relation of $\tilde{\Sigma}_{P_1P_2}$ by $\tilde{\Sigma}_{Q_1Q_2}$ if and only if for all $(z_{P_1P_2}, z_{Q_1Q_2}) \in S$,

1. for all $v_{P_1P_2} \in \mathcal{V}_{P_1P_2}^{\star}$ such that $E_{P_1P_2}v_{P_1P_2} = A_{P_1P_2}z_{P_1P_2}$ there should exist a $v_{Q_1Q_2} \in \mathcal{V}_{Q_1Q_2}^{\star}$ such that $E_{Q_1Q_2}v_{Q_1Q_2} = A_{Q_1Q_2}z_{Q_1Q_2}$ and $(v_{P_1P_2}, v_{Q_1Q_2}) \in \tilde{S}$
2. $C_{P_1P_2}z_{P_1P_2} = C_{Q_1Q_2}z_{Q_1Q_2}$.

The simulation relation $\tilde{S}$ is *full*, denoted by $\Sigma_{P_1P_2} \preccurlyeq \Sigma_{Q_1Q_2}$, if the projection on $\mathcal{Z}_{P_1P_2}$ equals the consistent subspace, that is, $\Pi_{P_1P_2}\tilde{S} = \mathcal{V}_{12}^{\star}$.

The linear algebraic characterization is derived similarly to Proposition 1 and Theorem 1.

Proposition 4. *There exists a simulation relation $S \subset \mathcal{X}_{P_1} \times \mathcal{X}_{P_2} \times \mathcal{X}_{Q_1} \times \mathcal{X}_{Q_2}$ of $\Sigma_{P_1} \|_{pc} \Sigma_{P_2}$ by $\Sigma_{Q_1} \|_{pc} \Sigma_{Q_2}$ if and only if for all $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$ and all $u \in \mathcal{U}_{P_1P_2}^{\star}$ the following holds:*

1. $\forall \begin{bmatrix} d_{P_1} \\ d_{P_2} \end{bmatrix} \in \mathcal{D}_1 \times \mathcal{D}_{P_2}\ \exists \begin{bmatrix} d_{Q_1} \\ d_{Q_2} \end{bmatrix} \in \mathcal{D}_{Q_1} \times \mathcal{D}_{Q_2}$: $\begin{bmatrix} A_{P_1}x_{P_1} + B_{P_1}u + L_{P_1}d_{P_1} \\ A_{P_2}x_{P_2} + B_{P_2}u + L_{P_2}d_{P_2} \\ A_{Q_1}x_{Q_1} + B_{Q_1}u + L_{Q_1}d_{Q_1} \\ A_{Q_2}x_{Q_2} + B_{Q_2}u + L_{Q_2}d_{Q_2} \end{bmatrix} \in S$
2. $C_{P_1}x_{P_1} = C_{P_2}x_{P_2} = C_{Q_1}x_{Q_1} = C_{Q_2}x_{Q_2}$.

Proof. With the system matrices (31), condition (2) in Definition 5 yields

$$u_{P_1} = u_{Q_1} \tag{35}$$

and

$$C_{P_1}x_{P_1} = C_{Q_1}x_{Q_1}. \tag{36}$$

Writing out condition (1) from Definition 5 results in

$$\begin{bmatrix} A_{P_1}x_{P_1} + B_{P_1}u_{P_1} + L_{P_1}d_{P_1} \\ A_{P_2}x_{P_2} + B_{P_2}u_{P_1} + L_{P_2}d_{P_2} \\ A_{Q_1}x_{Q_1} + B_{Q_1}u_{Q_1} + L_{Q_1}d_{Q_1} \\ A_{Q_2}x_{Q_2} + B_{Q_2}u_{Q_1} + L_{Q_2}d_{Q_2} \end{bmatrix} \in S \tag{37}$$

and

$$C_{P_1}x_{P_1} = C_{P_2}x_{P_2}, \qquad C_{Q_1}x_{Q_1} = C_{Q_2}x_{Q_2} \tag{38}$$

for all $(x_{P_1}, x_{P_2}, u_{P_1}, x_{Q_1}, x_{Q_2}, u_{Q_1}) \in \tilde{S}$ and

$$u_{P_1} \in \mathcal{U}_{P_1P_2}^{\star}. \tag{39}$$

Thus, Eqs. (35)–(39) are equivalent to the conditions (1) and (2) in Definition 5. □

Proposition 5. *There exists a simulation relation $S \subset \mathcal{X}_{P_1} \times \mathcal{X}_{P_2} \times \mathcal{X}_{Q_1} \times \mathcal{X}_{Q_2}$ of $\Sigma_{P_1} \|_{pc} \Sigma_{P_2}$ by $\Sigma_{Q_1} \|_{pc} \Sigma_{Q_2}$ if and only if the following conditions hold:*

1. *diag*$\{A_{P_1}, A_{P_2}, A_{Q_1}, A_{Q_2}\}S \subset S + \operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix}$

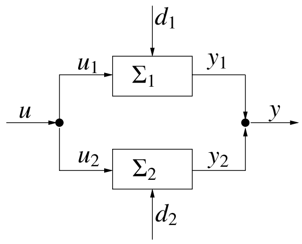

Fig. 4. $\Sigma_1 \|_{pc} \ \Sigma_2$.

2. $\operatorname{im} \begin{bmatrix} L_{P_1} & 0 \\ 0 & L_{P_2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \subset S + \operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix}$
3. $\operatorname{im} \begin{bmatrix} B_{P_1} \\ B_{P_2} \\ B_{Q_1} \\ B_{Q_2} \end{bmatrix} \cap (\mathcal{W}^\star_{P_1P_2} \times \mathcal{W}^\star_{Q_1Q_2}) \subset S + \operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix}$
4. $S \subset \ker \begin{bmatrix} C_{P_1} & -C_{P_2} & 0 & 0 \\ 0 & 0 & C_{Q_1} & -C_{Q_2} \\ C_{P_1} & 0 & -C_{Q_1} & 0 \end{bmatrix}$.

Proof. Condition (2) in Proposition 4 is equivalent to condition (4) in Proposition 5. Condition (1) in Proposition 4 results in

$$\operatorname{diag}\{A_{P_1}, A_{P_2}, A_{Q_1}, A_{Q_2}\}S + \operatorname{im} \begin{bmatrix} B_{P_1} \\ B_{P_2} \\ B_{Q_1} \\ B_{Q_2} \end{bmatrix} \subset S$$

$$+ \operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix}. \tag{40}$$

Since u is restricted to $u \in \mathcal{U}^\star_{P_1P_2}$, the image of the input map has to be restricted to the subspace of all admissible inputs, which is given by

$$\{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid \exists u : (x_{P_1}, x_{P_2}, u) \in \mathcal{V}^\star_{P_1P_2},$$
$$(x_{Q_1}, x_{Q_2}, u) \in \mathcal{V}^\star_{Q_1Q_2}\} = \mathcal{W}^\star_{P_1P_2} \times \mathcal{W}^\star_{Q_1Q_2}.$$

Therefore, conditions (1)–(3) in Proposition 5 are equivalent to condition (1) in Proposition 4. □

4.1. Compositional reasoning

We begin our analysis of parallel compositions by examining the compositionality property.

Theorem 5. *Given any four systems Σ_i, $i \in \{P_1, P_2, Q_1, Q_2\}$ of the form (27) respectively Fig. 4. Then parallel composition is compositional, i.e.*

$$\left.\begin{matrix} S_1\colon \Sigma_{P_1} \preccurlyeq \Sigma_{Q_1} \\ S_2\colon \Sigma_{P_2} \preccurlyeq \Sigma_{Q_2} \end{matrix}\right\} \Longrightarrow \Sigma_{P_1} \|_{pc} \ \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \|_{pc} \ \Sigma_{Q_2}. \tag{41}$$

Proof. Construct the relation S from given full simulation relations S_1 and S_2 of Σ_{P_1} and Σ_{P_2} by Σ_{Q_1}, respectively Σ_{Q_2}, as the product relation

$$S = \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid (x_{P_1}, x_{Q_1}) \in S_1, (x_{P_2}, x_{Q_2}) \in S_2\}.$$

Then for any $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$, any joint input $u \in \mathcal{U}^\star_{P_1P_2}$ and any d_{P_1}, d_{P_2} there exist d_{Q_1}, d_{Q_2} such that

$$\begin{bmatrix} A_{P_1}x_{P_1} + B_{P_1}u + G_{P_1}d_{P_1} \\ A_{P_2}x_{P_2} + B_{P_2}u + G_{P_2}d_{P_2} \\ A_{Q_1}x_{Q_1} + B_{Q_1}u + G_{Q_1}d_{Q_1} \\ A_{Q_2}x_{Q_2} + B_{Q_2}u + G_{Q_2}d_{Q_2} \end{bmatrix} \in S,$$

since for any d_{P_1} there exists a d_{Q_1} such that

$$\begin{bmatrix} A_{P_i}x_{P_i} + B_{P_i}u + G_{P_1}d_{P_1} \\ A_{Q_i}x_{Q_i} + B_{P_i}u + G_{Q_1}d_{Q_1} \end{bmatrix} \in S_i, \quad i = 1, 2 \tag{42}$$

for all $u \in \mathcal{U}$. Moreover, since $y_{P_1} = y_{Q_1}$ due to S_1 and $y_{P_2} = y_{Q_2}$ due to S_2 and $y_{P_1} = y_{P_2}$ as well as $y_{Q_1} = y_{Q_2}$ enforced by parallel composition, condition (2) in Proposition 4 is also fulfilled which proves that S is indeed a simulation relation of $\Sigma_{P_1} \|_{pc} \ \Sigma_{P_2}$ by $\Sigma_{Q_1} \|_{pc} \ \Sigma_{Q_2}$.
To show that S as defined in (41) is full, observe that (42) holds for all u. Since both S_1 and S_2 are full, we can find for every $u \in \mathcal{U}^\star_{P_1P_2}$ and every $(x_{P_1}, x_{P_2}) \in \mathcal{W}^\star_{P_1P_2}$ elements x_{Q_1}, x_{Q_2} such that $(x_{P_i}, x_{Q_i}) \in S_i$, $i = 1, 2$ and thus $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$. □

The converse is in general not true since the consistent subspace $\mathcal{V}^\star_{P_1P_2}$ restricts the choice of inputs u depending on the states x_{P_1}, x_{P_2}.

4.2. Decomposition of the specification

Throughout we have assumed that the given overall specification Σ_Q can be decomposed into sub-specifications Σ_{Q_i}, $i = 1, \ldots, N$, in the same way as the modeled system Σ_P consists of interconnected components Σ_{P_i}, $i = 1, \ldots, N$. For parallel compositions, the decomposition of the specification facilitates another deduction scheme to reduce the complexity of the verification task (8): In fact, we will show that fulfilment of the global specification Σ_Q is equivalent to fulfilling the individual sub-specifications Σ_{Q_i}. In other words, it is enough to prove that Σ_P is simulated by each of the sub-specifications Σ_{Q_i} to guarantee that it also fulfils the overall specification $\Sigma_Q = \Sigma_{Q_1} \|_{pc} \ \Sigma_{Q_2}$.

Proposition 6. *For any system Σ_P it holds that*

$$\Sigma_P \preccurlyeq \Sigma_P \|_{pc} \ \Sigma_P. \tag{43}$$

Proof. Construct a simulation relation S by setting all state variables to be the same,

$$S = \{(x_1, (x_2, x_3)) \mid x_1 = x_2 = x_3 \in \Sigma_P\}. \tag{44}$$

Then, S defines a full simulation relation of Σ_P by $\Sigma_P \| \Sigma_P$ since the evolution remains within the constrained subspace $Cx_1 = Cx_2 = Cx_3$ for all times. □

Proposition 7. *For any two systems Σ_P, Σ_Q, it holds that*

$$\Sigma_P \|_{pc} \ \Sigma_Q \preccurlyeq \Sigma_P. \tag{45}$$

Proof. The relation

$$S = \{((x_P, x_Q), \bar{x}_P) \mid x_P = \bar{x}_P, (x_P, x_Q) \in \mathcal{W}^\star_{PQ}\}$$

defines a full simulation relation of $\Sigma_P \|_{pc} \ \Sigma_Q$ by Σ_P. □

The main result to decompose a given global specification Σ_Q into an interconnection of local specifications Σ_{Q_1} and Σ_{Q_2} can be stated as follows.

Theorem 6. *Given a specification Σ_Q and systems Σ_{Q_i}, $i = 1, 2$, of the form (27). Then*

$$\Sigma_Q \preccurlyeq \Sigma_{Q_1} \|_{pc} \ \Sigma_{Q_2} \tag{46}$$

if and only if

$$\Sigma_Q \preccurlyeq \Sigma_{Q_1} \quad \text{and} \quad \Sigma_Q \preccurlyeq \Sigma_{Q_2}. \tag{47}$$

Proof. $\Longrightarrow$: Given a full simulation relation of Σ_Q by $\Sigma_{Q_1} \|_{pc} \ \Sigma_{Q_2}$, Proposition 7 allows us to conclude that

$$\Sigma_Q \preccurlyeq \Sigma_{Q_1}\|_{pc}\,\Sigma_{Q_2} \preccurlyeq \Sigma_{Q_1} \Longrightarrow \Sigma_Q \preccurlyeq \Sigma_{Q_1}$$

and by symmetry,

$$\Sigma_Q \preccurlyeq \Sigma_{Q_1}\|_{pc}\,\Sigma_{Q_2} \preccurlyeq \Sigma_{Q_2}\|_{pc}\,\Sigma_{Q_1} \preccurlyeq \Sigma_{Q_2} \Longrightarrow \Sigma_Q \preccurlyeq \Sigma_{Q_2}.$$

$\Longleftarrow$: Compositionality and Proposition 6 yield

$$\Sigma_Q\|_{pc}\,\Sigma_Q \preccurlyeq \Sigma_{Q_1}\|_{pc}\,\Sigma_{Q_2},\ \Sigma_Q \preccurlyeq \Sigma_Q\|_{pc}\,\Sigma_Q \Longrightarrow \Sigma_Q \preccurlyeq \Sigma_{Q_1}\|_{pc}\,\Sigma_{Q_2}.\quad \square$$

5. Outlook

The aim of this paper is to demonstrate that compositional techniques as used in computer science can help to simplify the analysis of complex control systems as well. The main results were obtained for linear systems interconnected either by feedback or parallel composition. The proposed methodology has the potential to be extended to other classes of systems; see [13] for initial results as to switching linear systems. Further generalizations could also be achieved using the more abstract framework presented in [8]. Besides, applications in the area of decentralized control are currently investigated. Another important direction of research is to investigate how to formulate system properties such as stability or controllability by means of simulations. Consider e.g. the problem of checking whether a linear system of the form

$$\Sigma: \quad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{48}$$

is lossless [16]. This can be reformulated as follows:
Σ is lossless if and only if there exists a simulation relation between Σ and the one-dimensional non-linear system

$$\Xi : \dot{\xi} = u^T y, \quad \xi \geq 0 \tag{49}$$

with external variables u and y. In fact, if $\frac{1}{2}x^T Qx$ represents a quadratic storage function for the system Σ, then the simulation relation S of Σ by Ξ is given by the graph

$$S = \left\{(x, \xi) \mid \xi = \frac{1}{2}x^T Qx\right\}. \tag{50}$$

Appendix A. Proof of Lemma 1

We give the proof with respect to $S_1 + \bar{S}_1$, the result for $S_2 + \bar{S}_2$ follows from symmetry. Take any $(x_{P_1}, \bar{x}_{Q_2}, x_{Q_1}, -x_{Q_2}) \in S_1$. Since all components fulfil

$$C_{P_1}x_{P_1} = C_{Q_1}x_{Q_1} = 0, \qquad C_{Q_2}x_{Q_2} = -C_{Q_2}\bar{x}_{Q_2} = 0 \tag{A.1}$$

and

$$H_{P_1}x_{P_1} = H_{Q_1}x_{Q_1} = 0, \qquad H_{Q_2}x_{Q_2} = -H_{Q_2}\bar{x}_{Q_2} = 0 \tag{A.2}$$

and condition (3) in Theorem 1 is fulfilled. By definition (6) there exists exists a $(x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in S_1$ and since S_1 is a simulation relation, condition (2) in Theorem 1 ensures that there exists a $(w_{P_1}, w_{Q_2}, w_{Q_1}, \bar{w}_{Q_2}) \in S_1$ such that

$$\begin{bmatrix} A_{P_1}x_{P_1} \\ A_{Q_2}x_{Q_2} \\ A_{Q_1}x_{Q_1} \\ A_{Q_2}\bar{x}_{Q_2} \end{bmatrix} = \begin{bmatrix} w_{P_1} \\ w_{Q_2} \\ w_{Q_1} \\ \bar{w}_{Q_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ L_{Q_1}\alpha \\ L_{Q_2}\beta \end{bmatrix}. \tag{A.3}$$

Note that since $(w_{P_1}, w_{Q_2}, w_{Q_1}, \bar{w}_{Q_2}) \in S_1$, $(w_{P_1}, \bar{w}_{Q_2}, w_{Q_1}, -w_{Q_2}) \in \bar{S}_1$. Hence

$$\begin{bmatrix} A_{P_1}x_{P_1} \\ A_{Q_2}\bar{x}_{Q_2} \\ A_{Q_1}x_{Q_1} \\ -A_{Q_2}x_{Q_2} \end{bmatrix} = \begin{bmatrix} w_{P_1} \\ \bar{w}_{Q_2} \\ w_{Q_1} \\ -w_{Q_2} \end{bmatrix} + \begin{bmatrix} 0 \\ L_{Q_2}\beta \\ L_{Q_1}\alpha \\ 0 \end{bmatrix}. \tag{A.4}$$

Since S_1 is a simulation relation, there exists for every $x \in \operatorname{im} G_{Q_2}$ an element $(0, x, \tilde{x}_{Q_1}, \tilde{x}_{Q_2}) \in S_1$ such that

$$\begin{bmatrix} 0 \\ x \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ x \\ \tilde{x}_{Q_1} \\ \tilde{x}_{Q_2} \end{bmatrix} + \operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix}. \tag{A.5}$$

Therefore, (A.4) can be rewritten as

$$\begin{bmatrix} A_{P_1}x_{P_1} \\ A_{Q_2}\bar{x}_{Q_2} \\ A_{Q_1}x_{Q_1} \\ -A_{Q_2}x_{Q_2} \end{bmatrix} = \underbrace{\begin{bmatrix} w_{P_1} \\ \bar{w}_{Q_2} \\ w_{Q_1} \\ -w_{Q_2} \end{bmatrix}}_{\in \bar{S}_1} + \underbrace{\begin{bmatrix} 0 \\ L_{Q_2}\beta \\ \tilde{x}_{Q_1} \\ \tilde{x}_{Q_2} \end{bmatrix}}_{\in S_1}$$

$$+ \operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix} \in S_1 + \bar{S}_1 + \operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix} \tag{A.6}$$

which proves that condition (2) in Theorem 1 is also fulfilled. Condition (1) is also fulfilled due to S_1 being a simulation relation. Indeed,

$$\operatorname{im} \begin{bmatrix} G_{P_1} & 0 \\ 0 & G_{Q_2} \\ G_{Q_1} & 0 \\ o & G_{Q_2} \end{bmatrix} + \operatorname{im} \begin{bmatrix} L_{P_1} & 0 \\ 0 & L_{Q_2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \subset S_1$$

$$+ \operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix} \subset S_1 + \bar{S}_1 + \operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix}. \tag{A.7}$$

Moreover, since S_1 is a full simulation relation, $\Pi_{X_{P_1}X_{Q_2}} S_1 = \Pi_{X_{P_1}X_{Q_2}}(S_1 + \bar{S}_1) = \mathcal{X}_{P_1} \times \mathcal{X}_{Q_2}$ and thus $S_1 + \bar{S}_1$ is a full simulation relation of $\Sigma_{P_1}\|\Sigma_{Q_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$.

Appendix B. Proof of Lemma 2

Again, the statement will be proved only for S_1^{sym}. Condition (1) and fullness of S_1^{sym} follow from fullness of S_1. Condition (3) holds true since by interchanging the components, still $C_{Q_2}x_{Q_2} = C_{Q_2}\bar{x}_{Q_2}$ as well as $H_{Q_2}x_{Q_2} = H_{Q_2}\bar{x}_{Q_2}$. Finally, condition (2) is proven analogously to (A.6) observing that for every $(x_{P_1}, \bar{x}_{Q_2}, x_{Q_1}, x_{Q_2}) \in \tilde{S}_1$, there exists a $(x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in S_1$ for which

$$\begin{bmatrix} A_{P_1}x_{P_1} + B_{P-1}C_{Q_2}x_{Q_2} \\ A_{Q_2}x_{Q_2} + B_{Q_2}C_{P_1}x_{P_1} \\ A_{Q_1}x_{Q_1} + B_{Q_1}C_{Q_2}\bar{x}_{Q_2} \\ A_{Q_2}\bar{x}_{Q_2} + B_{Q_2}C_{Q_1}x_{Q_1} \end{bmatrix} = \underbrace{\begin{bmatrix} w_{P_1} \\ w_{Q_2} \\ w_{Q_1} \\ \bar{w}_{Q_2} \end{bmatrix}}_{\in S_1} + \operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix}. \tag{B.1}$$

Furthermore, since S_1 is a simulation relation,

$$\operatorname{im} \begin{bmatrix} 0 \\ L_{Q_2} \\ 0 \\ 0 \end{bmatrix} \subset S_1 + \operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & _{Q_2} \end{bmatrix} \tag{B.2}$$

and therefore

$$\begin{bmatrix} A_{P_1} & B_{P_1}C_{Q_2} & 0 & 0 \\ B_{Q_2}C_{P_1} & A_{Q_2} & 0 & 0 \\ 0 & 0 & A_{Q_1} & B_{Q_1}C_{Q_2} \\ 0 & 0 & B_{Q_2}C_{Q_1} & A_{Q_2} \end{bmatrix} \tilde{S}_1$$

$$\subset S_1 + \tilde{S}_1 + \operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix}. \tag{B.3}$$

Appendix C. Proof of Lemma 3

Again, we will only prove the first half of the lemma. Since S_1 is a full simulation relation, it holds that for every $(0, x)$ there exists x_{Q_1}, x_{Q_2} such that $(0, x, x_{Q_1}, x_{Q_2}) \in S_1$ with $x_{Q_1} \in \ker C_{Q_1} \cap \ker H_{Q_1}$. If we take $x \in \ker C_{Q_2} \cap \ker H_{Q_2}$ then also $x_{Q_2} \in \ker C_{Q_2} \cap \ker H_{Q_2}$. Then $(0, x_{Q_2}, x_{Q_1}, -x) \in \bar{S}_1$ and therefore

$$\begin{bmatrix} 0 \\ x \\ x_{Q_1} \\ x_{Q_2} \end{bmatrix} - \begin{bmatrix} 0 \\ x_{Q_2} \\ x_{Q_1} \\ -x \end{bmatrix} = \begin{bmatrix} 0 \\ x - x_{Q_2} \\ 0 \\ x + x_{Q_2} \end{bmatrix} \in S_1 + \bar{S}_1. \tag{C.1}$$

Moreover, $(0, x+x_{Q_2}, 0, x-x_{Q_2}) \in (S_1+\bar{S}_1)^{\mathrm{sym}}$ and by the subspace property also

$$\begin{bmatrix} 0 \\ x - x_{Q_2} \\ 0 \\ x + x_{Q_2} \end{bmatrix} + \begin{bmatrix} 0 \\ x + x_{Q_2} \\ 0 \\ x - x_{Q_2} \end{bmatrix} = 2 \begin{bmatrix} 0 \\ x \\ 0 \\ x \end{bmatrix} \in (S_1 + \bar{S}_1)^{\mathrm{sym}}. \tag{C.2}$$

Appendix D. Proof of Theorem 4

Firstly, it is easy to see that S indeed defines a linear subspace. Secondly, we have to show that it defines a simulation relation of $\Sigma_{P_1} \| \Sigma_{P_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$. Take any $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$. Then there exist $(x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in (S_1 + \bar{S}_1)^{\mathrm{sym}}$ and $(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \in (S_2 + \bar{S}_2)^{\mathrm{sym}}$ with the property that

$$C_{P_1} x_{P_1} = C_{Q_1} x_{Q_1} = C_{Q_1} \bar{x}_{Q_1}, \qquad C_{P_2} x_{P_2} = C_{Q_2} x_{Q_2} = C_{Q_2} \bar{x}_{Q_2} \tag{D.1}$$

and

$$H_{P_1} x_{P_1} = H_{Q_1} x_{Q_1} = H_{Q_1} \bar{x}_{Q_1}, \qquad H_{P_2} x_{P_2} = H_{Q_2} x_{Q_2} = H_{Q_2} \bar{x}_{Q_2} \tag{D.2}$$

so that condition (3) of Theorem 1 is already fulfilled. To show that condition (1) also holds, take first any $d_{P_1} \in \operatorname{im} L_{P_1}$. Since $(S_1 + \bar{S}_1)^{\mathrm{sym}}$ is a simulation relation, there exist $x_{Q_i} \in \operatorname{im} L_{Q_i}$, $i = 1, 2$ such that

$$\begin{bmatrix} d_{P_1} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} d_{P_1} \\ 0 \\ x_{Q_1} \\ x_{Q_2} \end{bmatrix} + \operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix} \tag{D.3}$$

with $(d_{P_1}, 0, x_{Q_1}, x_{Q_2}) \in (S_1 + \bar{S}_1)^{\mathrm{sym}}$. Since $x_{Q_1} \in \operatorname{im} G_{Q_1}$ and $(S_2+\bar{S}_2)^{\mathrm{sym}}$ is also a full simulation relation, there exist $\bar{x}_{Q_1} \in \operatorname{im} L_{Q_1}$ and $\bar{x}_{Q_2} \in \operatorname{im} L_{Q_2} \cap \ker C_{Q_2} \cap \ker H_{Q_1}$ such that $(x_{Q_1}, 0, \bar{x}_{Q_1}, \bar{x}_{Q_2}) \in (S_2 + \bar{S}_2)^{\mathrm{sym}}$. By Lemma 3, there exists an element $(0, \bar{x}_{Q_2}, 0, \bar{x}_{Q_2}) \in (S_1 + \bar{S}_1)^{\mathrm{sym}}$ and therefore

$$\begin{gathered} \begin{bmatrix} d_{P_1} \\ 0 \\ x_{Q_1} \\ x_{Q_2} \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{x}_{Q_2} \\ 0 \\ \bar{x}_{Q_2} \end{bmatrix} = \begin{bmatrix} d_{P_1} \\ \bar{x}_{Q_2} \\ x_{Q_1} \\ x_{Q_2} + \bar{x}_{Q_2} \end{bmatrix} \in (S_1 + \bar{S}_1)^{\mathrm{sym}}, \\ \begin{bmatrix} x_{Q_1} \\ 0 \\ \bar{x}_{Q_1} \\ \bar{x}_{Q_2} \end{bmatrix} \in (S_2 + \bar{S}_2)^{\mathrm{sym}} \Longrightarrow \begin{bmatrix} d_{P_1} \\ 0 \\ x_{Q_1} \\ \bar{x}_{Q_2} \end{bmatrix} \in S \end{gathered} \tag{D.4}$$

for any $d_{P_1} \in \operatorname{im} L_{P_1}$ with $x_{Q_1} \in \operatorname{im} L_{Q_1}$ and $\bar{x}_{Q_2} \in \operatorname{im} L_{Q_2} \cap \ker C_{Q_2} \cap \ker H_{Q_2}$. Hence,

$$\operatorname{im} \begin{bmatrix} L_{P_1} \\ 0 \\ 0 \\ 0 \end{bmatrix} \subset S + \operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix}. \tag{D.5}$$

By the same arguments one can also show that

$$\operatorname{im} \begin{bmatrix} 0 \\ L_{P_2} \\ 0 \\ 0 \end{bmatrix} \subset S + \operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix}. \tag{D.6}$$

Similarly, consider any

$$\begin{bmatrix} g_{P_1} \\ 0 \\ g_{Q_1} \\ 0 \end{bmatrix} \in \operatorname{im} \begin{bmatrix} G_{P_1} \\ 0 \\ G_{Q_1} \\ 0 \end{bmatrix} \subset (S_1 + \bar{S}_1)^{\mathrm{sym}} + \operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix}. \tag{D.7}$$

From (D.7) it follows that there exists an element $(x_{P_1}, 0, x_{Q_1}, x_{Q_2}) \in (S_1 + \bar{S}_1)^{\mathrm{sym}}$ such that

$$\begin{bmatrix} g_{P_1} \\ 0 \\ g_{Q_1} \\ 0 \end{bmatrix} = \begin{bmatrix} x_{P_1} \\ 0 \\ x_{Q_1} \\ x_{Q_2} \end{bmatrix} + \operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix} \tag{D.8}$$

with $x_{Q_2} \in \operatorname{im} L_{Q_2} \cap \ker C_{Q_2} \cap \ker H_{Q_2}$. Since $(S_2 + \bar{S}_2)^{\mathrm{sym}}$ is full, there exists an element $(x_{Q_1}, 0, \bar{x}_{Q_1}, \bar{x}_{Q_2}) \in (S_2 + \bar{S}_2)^{\mathrm{sym}}$ such that $\bar{x}_{Q_2} \in \operatorname{im} L_{Q_2} \cap \ker C_{Q_2} \cap \ker H_{Q_2}$. Lemma 3 ensures that there also exists an element $(0, \bar{x}_{Q_2}, 0, \bar{x}_{Q_2}) \in (S_1 + \bar{S}_1)^{\mathrm{sym}}$ since $x_{Q_2}, \bar{x}_{Q_2}$ and therefore $\bar{x}_{Q_2} \in \operatorname{im} L_{Q_2} \ker C_{Q_2} \cap \ker H_{Q_2}$ as well as

$$\begin{bmatrix} x_{P_1} \\ 0 \\ x_{Q_1} \\ x_{Q_2} \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{x}_{Q_2} \\ 0 \\ \bar{x}_{Q_2} \end{bmatrix} = \begin{bmatrix} x_{P_1} \\ \bar{x}_{Q_2} \\ x_{Q_1} \\ \bar{x}_{Q_2} + x_{Q_2} \end{bmatrix} \in (S_1 + \bar{S}_1)^{\mathrm{sym}}. \tag{D.9}$$

Therefore, there exists an element $(x_{P_1}, 0, x_{Q_1}, \bar{x}_{Q_2}) \in S$ with $\bar{x}_{Q_2} \in \operatorname{im} L_{Q_2}$ such that

$$\begin{bmatrix} g_{P_1} \\ 0 \\ g_{Q_1} \\ 0 \end{bmatrix} = \begin{bmatrix} x_{P_1} \\ 0 \\ x_{Q_1} \\ \bar{x}_{Q_2} \end{bmatrix} + \operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix} \tag{D.10}$$

which proves that

$$\operatorname{im} \begin{bmatrix} G_{P_1} \\ 0 \\ G_{Q_1} \\ 0 \end{bmatrix} \subset S + \operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix}. \tag{D.11}$$

Similarly, one can show that

$$\operatorname{im} \begin{bmatrix} 0 \\ G_{P_2} \\ 0 \\ G_{Q_2} \end{bmatrix} \subset S + \operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix} \tag{D.12}$$

and therefore condition (1) in Theorem 1 is completely fulfilled. As to condition (2), take any $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$. Since $(S_i + \bar{S}_i)^{\mathrm{sym}}$, $i = 1, 2$ are simulation relations, there exist $(x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}), (v_{P_1}, v_{Q_2}, v_{Q_1}, \bar{v}_{Q_2}) \in (S_1 + \bar{S}_1)^{\mathrm{sym}}$ and $a, b \in \mathbb{R}$ such that

$$\begin{bmatrix} A_{P_1} x_{P_1} + B_{P_1} C_{Q_2} x_{Q_2} \\ A_{Q_2} x_{Q_2} + B_{Q_2} C_{P_1} x_{P_1} \\ A_{Q_1} x_{Q_1} + B_{Q_1} C_{Q_2} \bar{x}_{Q_2} \\ A_{Q_2} \bar{x}_{Q_2} + B_{Q_1} C_{Q_1} x_{Q_1} \end{bmatrix} = \begin{bmatrix} v_{P_1} \\ v_{Q_2} \\ v_{Q_1} \\ \bar{v}_{Q_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ L_{Q_1} a \\ L_{Q_2} b \end{bmatrix} \tag{D.13}$$

as well as $(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}), (w_{Q_1}, w_{P_2}, \bar{w}_{Q_1}, w_{Q_2}) \in (S_2 + \bar{S}_2)^{\mathrm{sym}}$ and $l, m \in \mathbb{R}$ such that

$$\begin{bmatrix} A_{Q_1} x_{Q_1} + B_{Q_1} C_{P_2} x_{P_2} \\ A_{P_2} x_{P_2} + B_{P_2} C_{Q_1} x_{Q_1} \\ A_{Q_1} \bar{x}_{Q_1} + B_{Q_1} C_{Q_2} x_{Q_2} \\ A_{Q_2} x_{Q_2} + B_{Q_1} C_{Q_1} \bar{x}_{Q_1} \end{bmatrix} = \begin{bmatrix} w_{Q_1} \\ w_{P_2} \\ \bar{w}_{Q_1} \\ w_{Q_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ L_{Q_1} l \\ L_{Q_2} m \end{bmatrix}. \tag{D.14}$$

Because of (D.1), observe that $v_{Q_2} = w_{Q_2} + L_{Q_2}m$ and $v_{Q_1} + L_{Q_1}a = w_{Q_1}$. Furthermore, we know that there exists an element $(0, L_{Q_2}m, L_{Q_1}c, L_{Q_2}d) \in (S_1 + \bar{S}_1)^{\mathrm{sym}}$ with $L_{Q_1}c \in \ker C_{Q_1} \cap \ker H_{Q_1}$ and similarly, $(L_{Q_1}a, 0, L_{Q_1}n, L_{Q_2}p) \in (S_2 + \bar{S}_2)^{\mathrm{sym}}$ with $L_{Q_2}p \in \ker C_{Q_2} \cap \ker H_{Q_2}$. With Lemma 3, also $(0, L_{Q_2}p, 0, L_{Q_2}p) \in (S_1 + \bar{S}_1)^{\mathrm{sym}}$ and $(L_{Q_1}c, 0, L_{Q_1}c, 0) \in (S_2 + \bar{S}_2)^{\mathrm{sym}}$. Hence,

$$\begin{bmatrix} v_{P_1} \\ v_{Q_2} \\ v_{Q_1} \\ \bar{v}_{Q_2} \end{bmatrix} = \begin{bmatrix} v_{P_1} \\ v_{Q_2} - L_{Q_2}m - L_{Q_2}p \\ v_{Q_1} - L_{Q_1}c \\ \bar{v}_{Q_2} - L_{Q_2}d - L_{Q_2}p \end{bmatrix} + \begin{bmatrix} 0 \\ L_{Q_2}(m+p) \\ L_{Q_1}c \\ L_{Q_2}(d+p) \end{bmatrix}$$
$$= \begin{bmatrix} v_{P_1} \\ w_{Q_2} - L_{Q_2}p \\ v_{Q_1} - L_{Q_1}c \\ \bar{v}_{Q_2} - L_{Q_2}(d+p) \end{bmatrix} + \begin{bmatrix} 0 \\ L_{Q_2}(m+p) \\ L_{Q_1}c \\ L_{Q_2}(d+p) \end{bmatrix} \tag{D.15}$$

and

$$\begin{bmatrix} w_{Q_1} \\ w_{P_2} \\ \bar{w}_{Q_1} \\ w_{Q_2} \end{bmatrix} = \begin{bmatrix} w_{Q_1} - L_{Q_1}a - L_{Q_1}c \\ w_{P_2} \\ \bar{w}_{Q_1} - L_{Q_1}n - L_{Q_1}c \\ w_{Q_2} - L_{Q_2}p \end{bmatrix} + \begin{bmatrix} L_{Q_1}(a+c) \\ 0 \\ L_{Q_1}(n+c) \\ L_{Q_2}p \end{bmatrix}$$
$$= \begin{bmatrix} v_{Q_1} - L_{Q_1}c \\ w_{P_2} \\ \bar{w}_{Q_1} - L_{Q_1}(n+c) \\ w_{Q_2} - L_{Q_2}p \end{bmatrix} + \begin{bmatrix} L_{Q_1}(a+c) \\ 0 \\ L_{Q_1}(n+c) \\ L_{Q_2}p \end{bmatrix}. \tag{D.16}$$

Thus, (D.13) can be rewritten as

$$\begin{bmatrix} A_{P_1}x_{P_1} + B_{P_1}C_{Q_2}x_{Q_2} \\ A_{Q_2}x_{Q_2} + B_{Q_2}C_{P_1}x_{P_1} \\ A_{Q_1}x_{Q_1} + B_{Q_1}C_{Q_2}\bar{x}_{Q_2} \\ A_{Q_2}\bar{x}_{Q_2} + B_{Q_1}C_{Q_1}x_{Q_1} \end{bmatrix} = \begin{bmatrix} v_{P_1} \\ w_{Q_2} - L_{Q_2}p \\ v_{Q_1} - L_{Q_1}c \\ \bar{v}_{Q_2} - L_{Q_2}(d+p) \end{bmatrix} + \begin{bmatrix} 0 \\ L_{Q_2}(m+p) \\ L_{Q_1}(c+a) \\ L_{Q_2}(d+p+b) \end{bmatrix} \tag{D.17}$$

and similarly, (D.14) becomes

$$\begin{bmatrix} A_{Q_1}x_{Q_1} + B_{Q_1}C_{P_2}x_{P_2} \\ A_{P_2}x_{P_2} + B_{P_2}C_{Q_1}x_{Q_1} \\ A_{Q_1}\bar{x}_{Q_1} + B_{Q_1}C_{Q_2}x_{Q_2} \\ A_{Q_2}x_{Q_2} + B_{Q_1}C_{Q_1}\bar{x}_{Q_1} \end{bmatrix} = \begin{bmatrix} v_{Q_1} - L_{Q_1}c \\ w_{P_2} \\ \bar{w}_{Q_1} - L_{Q_1}(n+c) \\ w_{Q_2} - L_{Q_2}p \end{bmatrix} + \begin{bmatrix} L_{Q_1}(a+c) \\ 0 \\ L_{Q_1}(n+c+l) \\ L_{Q_2}(p+m) \end{bmatrix}. \tag{D.18}$$

Consequently, there exists an element $(v_{P_1}, w_{P_2}, v_{Q_1} - L_{Q_1}c, w_{Q_2} - L_{Q_2}p) \in S$ such that

$$\begin{bmatrix} A_{P_1}x_{P_1} + B_{P_1}C_{P_2}x_{P_2} \\ A_{P_2}x_{P_2} + B_{P_2}C_{P_1}x_{P_1} \\ A_{Q_1}x_{Q_1} + B_{Q_1}C_{Q_2}x_{Q_2} \\ A_{Q_2}x_{Q_2} + B_{Q_1}C_{Q_1}x_{Q_1} \end{bmatrix} = \begin{bmatrix} v_{P_1} \\ w_{P_2} \\ v_{Q_1} - L_{Q_1}c \\ w_{Q_2} - L_{Q_2}p \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ L_{Q_1}(a+c) \\ L_{Q_2}(p+m) \end{bmatrix} \tag{D.19}$$

which concludes the proof for S being a simulation relation of $\Sigma_{P_1} \| \Sigma_{P_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$.

Thirdly, it has to be shown that S as defined in (11) is full, i.e. for any (x_{P_1}, x_{P_2}) there has to exist a (x_{Q_1}, x_{Q_2}) such that $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$. Since $(S_1 + \bar{S}_1)^{\mathrm{sym}}$ is a full simulation relation, there exists for every (x_{P_1}, x_{Q_2}) a $(\bar{x}_{Q_1}, \bar{x}_{Q_1})$ such that $(x_{P_1}, x_{Q_2}, \bar{x}_{Q_1}, \bar{x}_{Q_1}) \in (S_1 + \bar{S}_1)^{\mathrm{sym}}$. Moreover, since $(S_2 + \bar{S}_2)^{\mathrm{sym}}$ is also full, there exists for an arbitrary x_{P_2} and the given $\bar{x}_{Q_1}$ a $(\hat{x}_{Q_1}, \hat{x}_{Q_2})$ such that $(\bar{x}_{Q_1}, x_{P_2}\hat{x}_{Q_1}, \hat{x}_{Q_2}) \in (S_2 + \bar{S}_2)^{\mathrm{sym}}$. Fullness of $(S_1 + \bar{S}_1)^{\mathrm{sym}}$ also ensures that there exists an element $(0, \hat{x}_{Q_2}, \tilde{x}_{Q_1}, \tilde{x}_{Q_2}) \in (S_1 + \bar{S}_1)^{\mathrm{sym}}$ with $\tilde{x}_{Q_1} \in \ker C_{Q_1} \cap \ker H_{Q_1}$. By Lemma 3, however, an element $(\tilde{x}_{Q_1}, 0, \tilde{x}_{Q_1}, 0)$ is contained in $(S_2 + \bar{S}_2)^{\mathrm{sym}}$. Hence

$$\begin{bmatrix} x_{P_1} \\ x_{Q_2} \\ \bar{x}_{Q_1} \\ \bar{x}_{Q_2} \end{bmatrix} + \begin{bmatrix} 0 \\ \hat{x}_{Q_2} - x_{Q_2} \\ \tilde{x}_{Q_1} \\ \tilde{x}_{Q_2} \end{bmatrix} = \begin{bmatrix} x_{P_1} \\ \hat{x}_{Q_2} \\ \bar{x}_{Q_1} + \tilde{x}_{Q_1} \\ \bar{x}_{Q_2} + \tilde{x}_{Q_2} \end{bmatrix} \in (S_1 + \bar{S}_1)^{\mathrm{sym}} \tag{D.20}$$
$$\begin{bmatrix} \bar{x}_{Q_1} \\ x_{P_2} \\ \hat{x}_{Q_1} \\ \hat{x}_{Q_2} \end{bmatrix} + \begin{bmatrix} \tilde{x}_{Q_1} \\ 0 \\ \tilde{x}_{Q_1} \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{x}_{Q_1} + \tilde{x}_{Q_1} \\ x_{P_2} \\ \hat{x}_{Q_1} + \tilde{x}_{Q_1} \\ \hat{x}_{Q_2} \end{bmatrix} \in (S_2 + \bar{S}_2)^{\mathrm{sym}}$$

from which the element

$$\begin{bmatrix} x_{P_1} \\ x_{P_2} \\ \hat{x}_{Q_1} + \tilde{x}_{Q_1} \\ \hat{x}_{Q_2} \end{bmatrix} \in S \tag{D.21}$$

can be constructed for any (x_{P_1}, x_{P_2}).

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